题：let $f(x)=e^{-\frac{1}{x^2}}$ for $x\neq 0$ and $f(0)=0$, prove that f is infinite differentiable at 0. with $f^{(n)}(0)=0$ for all $n\in N$. You can use anything you learned in calculus. e.g. the derivative of e^x, the derivative of $\log x$, etc.

solution: $n\geq 1$, $\lim\limits_{x\to\infty} x^n e^{-x^2} = \lim\limits_{x\to\infty}\frac{x^n}{e^{x^2}} = \lim\limits_{x\to\infty}\frac{nx^{n-1}}{2xe^{x^2}}$

$$=\lim_{x\to\infty}\frac{nx^{n-2}}{2e^{x^2}} = \cdots\cdots = \lim_{x\to\infty}\frac{n(n-2)(n-4)\cdots 1}{2^{[\frac{n}{2}]}e^{x^2}} = 0$$

let $R(x)=ax^n$, $(a\in R, n\geq 1)$ $\lim\limits_{x\to\infty} R(x)e^{-x^2}=0$

let $\Delta x=\frac{1}{x}$ the $\lim\limits_{\Delta x\to 0}\frac{a}{\Delta x^n}e^{-\frac{1}{\Delta x^2}}=0$

$$f'(0)=\lim_{\Delta x\to 0}\frac{e^{-\frac{1}{\Delta x^2}}-0}{\Delta x}=\lim_{\Delta x\to 0}\frac{e^{-\frac{1}{\Delta x^2}}}{\Delta x}=\lim_{\Delta x\to 0}\frac{\frac{1}{\Delta x^3}e^{-\frac{1}{\Delta x^2}}}{1}=0$$

$x\neq 0$ $f'(x)=\frac{2}{x^3}e^{-\frac{1}{x^2}}$

$$f''(0)=\lim_{\Delta x\to 0}\frac{f'(0+\Delta x)-f'(0)}{\Delta x}=\lim_{\Delta x\to 0}\frac{2e^{-\frac{1}{x^2}}}{\Delta x^4}=0$$

$\vdots$

$$f^{(n)}(0)=\lim_{\Delta x\to 0}\frac{f^{(n-1)}(0+\Delta x)-f^{(n-1)}(0)}{\Delta x}=\lim_{x\to\infty}R(x)e^{-x^2}=0$$

above all $f(x)=\begin{cases}e^{-\frac{1}{x^2}}, & x\neq 0\\ 0, & x=0\end{cases}$ is infinite differentiable at 0. with $f^{(n)}(0)=0$ for $n\in N$。