题：Let x_1, x_2 be the values of a random sample of size 2 from a Uniform[0,β] distribution.

(a) Find k so that $0<\beta<k(x_1+x_2)$ is a (1-α)100% confidence interval when $\alpha \leqq 1/2$.

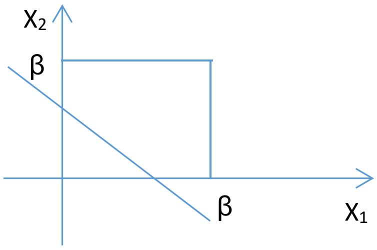
(b) Find k so that $0<\beta<k(x_1+x_2)$ is a (1-α)100% confidence interval when $\alpha > 1/2$.

(c) The range R of the sample is defined as $R=X_{(2)}-X_{(1)}$. You can use without proof the fact that the density of R is given by $f_R(r)=\begin{cases}\frac{2}{\beta^2}(\beta-r), 0<r<\beta \\ 0, otherwise\end{cases}$ Use this result to find c so that $R<\beta<cR$ is a (1-α)100% confidence interval for β

解：∵ x_1, x_2 be the values of a random sample of size 2 from a Uniform[0,β] distribution.

$$\therefore f_1(x_1)=\begin{cases}1/\beta, 0<x_1<\beta \\ 0, otherwise\end{cases}; f_2(x_2)=\begin{cases}1/\beta, 0<x_2<\beta \\ 0, otherwise\end{cases}; f(x_1,x_2)=\begin{cases}1/\beta^2, 0<x_1,x_2<\beta \\ 0, otherwise\end{cases}$$

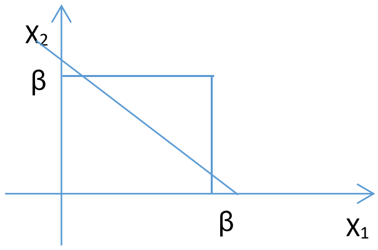
$P(0<\beta<k(x_1+x_2))=p(\beta<k(x_1+x_2))-p(\beta\leqq 0)=p(\beta<k(x_1+x_2))=(1-\alpha)100\%$



(a) $\alpha\leqq 1/2$，$p(\beta<k(x_1+x_2))=P(x_1+x_2>\beta/k)=(1-\alpha)100\%$

如左图斜线上部面积为，$\beta^2-(1/2)(\beta/k)^2$

$P(x_1+x_2>\beta/k)=[\beta^2-(1/2)(\beta/k)^2]/\beta^2=1-1/(2k^2)=1-\alpha$, $k=\sqrt{\frac{1}{2\alpha}}$



(b) $\alpha>1/2$ $p(\beta<k(x_1+x_2))=P(x_1+x_2>\beta/k)=(1-\alpha)100\%$

如左图斜线上部面积为，$(1/2)(2\beta-\beta/k)^2$

$P(x_1+x_2>\beta/k)=(1/2)(2\beta-\beta/k)^2/\beta^2=(1/2)(2-1/k)^2=1-\alpha$, $k=\frac{1}{2-\sqrt{2-2\alpha}}$

(c) $P(R<\beta<cR)=P(\beta<cR)-P(R<\beta)=P(\beta/c<R<\beta)=(1-\alpha)100\%$,

$$P(\beta/c<R<\beta)=\int_{\beta/c}^{\beta}\frac{2}{\beta^2}(\beta-r)dr=-\frac{1}{\beta^2}(\beta-r)^2\Big|_{\beta/c}^{\beta}=(1-\frac{1}{c})^2=1-\alpha$$

$$C=\frac{1}{1-\sqrt{1-\alpha}}$$